ON STANLEY'S RECIPROCITY THEOREM FOR RATIONAL CONES

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Abstract. We give a short, self-contained proof of Stanley's reciprocity theorem for a rational cone $\mathcal{K} \subset \mathbb{R}^d$. Namely, let $\sigma_{\mathcal{K}}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$. Then $\sigma_{\mathcal{K}}(\mathbf{x})$ and $\sigma_{\mathcal{K}^\circ}(\mathbf{x})$ are rational functions which satisfy the identity $\sigma_{\mathcal{K}}(1/\mathbf{x}) = (-1)^d \sigma_{\mathcal{K}^\circ}(\mathbf{x})$. A corollary of Stanley's theorem is the Ehrhart-Macdonald reciprocity theorem for the lattice-point enumerator of rational polytopes. A distinguishing feature of our proof is that it uses neither the shelling of a polyhedron nor the concept of finite additive measures. The proof follows from elementary techniques in contour integration.

1. Introduction

Let $\mathcal{K}$ be a *rational cone*, that is, of the form

$$\mathcal{K} = \left\{ \mathbf{r} \in \mathbb{R}^d : \ \mathbf{A}\,\mathbf{r} \le 0 \right\}$$

for some integral $m \times d$-matrix $\mathbf{A}$. Thus $\mathcal{K}$ is a real cone in $\mathbb{R}^d$ defined by m inequalities. We assume that $\mathcal{K}$ is d-dimensional and pointed: i.e., that $\mathcal{K}$ does not contain a line.

In his study of nonnegative integral solutions to linear systems, Stanley was led to consider the generating functions

$$\sigma_{\mathcal{K}}(\mathbf{x}) = \sum_{\mathbf{m} \in \mathcal{K} \cap \mathbb{Z}^d} \mathbf{x}^{\mathbf{m}}$$

and its companion $\sigma_{\mathcal{K}^\circ}(\mathbf{x})$ for the interior $\mathcal{K}^\circ$ of $\mathcal{K}$. Here $\mathbf{x}^{\mathbf{m}}$ denotes the product $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$. The function $\sigma_{\mathcal{K}}$ (as well as $\sigma_{\mathcal{K}^\circ}$) is a rational function in the components of $\mathbf{x}$; this is proved, for example, by triangulating $\mathcal{K}$ into simplicial cones, for which one can explicitly form the rational functions representing their generating functions (see, for example, [9, Chapter 4]). The fundamental reciprocity theorem of Stanley [7] relates the two rational functions $\sigma_{\mathcal{K}}$ and $\sigma_{\mathcal{K}^\circ}$. We abbreviate the vector $(1/x_1, 1/x_2, \ldots, 1/x_d)$ by $1/\mathbf{x}$.

Theorem 1 (Stanley). *As rational functions,* $\sigma_{\mathcal{K}}(\mathbf{x}) = (-1)^d \sigma_{\mathcal{K}^\circ}(1/\mathbf{x})$.

The proof of Stanley's theorem is not quite as simple as that of the rationality of $\sigma_{\mathcal{K}}$. Most proofs of Theorem 1 that we are aware use either the concept of shelling of a polyhedron or the concept of finitely additive measures (also called valuations). Both concepts reduce the theorem to simplicial cones, for which Theorem 1 is not hard to prove. In this paper, we give a proof which

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neither depends on shelling nor on valuations. It is based on rational generating functions, which essentially go back to Euler. In [7] Stanley used these generating functions to suggest a residue-computation approach to proving Theorem 1; he then implemented it by computing each of the residues in gory detail. Here we show that one can, in fact, reduce Theorem 1 to a simple change of variables in the integral over the generating function, proving Stanley's reciprocity theorem without any actual computation of residues or other technical machinery.

2. Euler's generating function

Suppose we have an element ϕ of $SL_d(\mathbb{Z})$, i.e. a unimodular transformation preserving the integer lattice. We claim that the image $\phi(\mathcal{K})$ will satisfy the reciprocity law if and only if $\mathcal{K}$ does. Indeed, it is easy to see that the generating function $\sigma_{\phi(\mathcal{K})}$ is obtained by substituting $\mathbf{x}^{\phi(e_i)}$ for x_i (for all i) in the generating function $\sigma_{\mathcal{K}}$, and similarly the generating function $\sigma_{\phi(\mathcal{K}^\circ)}$ is obtained from $\sigma_{\mathcal{K}^\circ}$ by the same substitutions. From this observation, we immediately obtain that if $\sigma_{\mathcal{K}}(\mathbf{x}) = \sigma_{\mathcal{K}^\circ}(1/\mathbf{x})$, then

$$\sigma_{\phi(\mathcal{K})}(\mathbf{x}) = \sigma_{\mathcal{K}}\left(\mathbf{x}^{\phi(e_i)}\right) = \sigma_{\mathcal{K}^\circ}\left(\mathbf{x}^{-\phi(e_i)}\right) = \sigma_{\phi(\mathcal{K}^\circ)}(1/\mathbf{x}),$$

so $\sigma_{\phi(\mathcal{K})}$ and $\sigma_{\phi(\mathcal{K}^\circ)}$ also satisfy the reciprocity law.

Now, given any pointed cone $\mathcal{K}$, we can find a unimodular transformation ϕ so that $\phi(\mathcal{K})$ lies in the nonnegative orthant, and intersects its boundary only at $(0, 0, \ldots, 0)$: simply pick a lattice basis such that $\mathcal{K}$ is contained in the interior of the cone it spans, and send this to the standard basis. (This is easy to do; for instance, take any hyperplane which intersects $\mathcal{K}$ only at the origin, pick a lattice basis for the sublattice contained in it, and take the final point at lattice distance 1 in the direction of $\mathcal{K}$, and very far away in the direction opposite the $d-1$ points in the hyperplane.) Therefore, it suffices to consider the case where $\mathcal{K} \subset \mathbb{R}^d_{\geq 0}$ and $(0, 0, \ldots, 0)$ is the only point in $\mathcal{K}$ with any zero coordinate. We will assume that $\mathcal{K}$ has this form from now on.

Denote the columns of $\mathbf{A}$ by $\mathbf{a}_1, \ldots, \mathbf{a}_d$. Our main tool for proving Theorem 1 is the following lemma, the idea of which goes back to at least Euler [3] and which is proved by simply expanding geometric series.

Lemma 2. $\sigma_{\mathcal{K}}(\mathbf{x})$ *equals the constant* $\mathbf{z}$*-coefficient of the function*

$$\frac{1}{(1 - x_1\mathbf{z}^{\mathbf{a}_1})\,(1 - x_2\mathbf{z}^{\mathbf{a}_2}) \cdots (1 - x_d\mathbf{z}^{\mathbf{a}_d})\,(1 - z_1)\,(1 - z_2) \cdots (1 - z_m)}$$

expanded as a power series centered at $\mathbf{z} = 0$*.*

We will use an integral version of this lemma, for which we need additional variables to avoid integrating over singularities. Let

$$\theta_{\mathcal{K}}(\mathbf{x}, \mathbf{y}) = \theta_{\mathcal{K}}\left(x_1, \ldots, x_d, y_1, \ldots, y_m\right) = \mathrm{const}_{\mathbf{z}}\left(\frac{1}{\prod_{j=1}^d \left(1 - x_j\mathbf{z}^{\mathbf{a}_j}\right) \prod_{k=1}^m \left(1 - y_k z_k\right)}\right). \tag{1}$$

This is a rational function in the coordinates of $\mathbf{x}$ and $\mathbf{y}$, and $\sigma_{\mathcal{K}}(\mathbf{x}) = \theta_{\mathcal{K}}(\mathbf{x}, 1, 1, \ldots, 1)$. The y-variables represent slack variables, one for each inequality; thus instead of $\mathbf{A}\,\mathbf{x} \leq 0$ and $\mathbf{x} \geq 0$, we consider the cone given by $(\mathbf{A} \,|\, I) \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = 0$ and $\mathbf{x}, \mathbf{y} \geq 0$, where I is the n-by-n identity

matrix. We next translate this constant-term identity into an integral identity. We abbreviate the measure $\frac{dz_1}{z_1}\frac{dz_2}{z_2}\cdots\frac{dz_m}{z_m}$ by $\frac{d\mathbf{z}}{\mathbf{z}}$, and we write $|\mathbf{x}|<1$ to indicate that $|x_1|,|x_2|,\dots,|x_d|<1$. Then for $|\mathbf{x}|,|\mathbf{y}|<1$

$$\theta_{\mathcal{K}}(\mathbf{x},\mathbf{y})=\int\frac{1}{\prod_{j=1}^{d}\left(1-x_j\mathbf{z}^{\mathbf{a}_j}\right)\prod_{k=1}^{m}\left(1-y_kz_k\right)}\frac{d\mathbf{z}}{\mathbf{z}}\ ,$$

where the integral sign stands for $1/(2\pi i)^m$ times an m-fold integral, each one over the unit circle. Note that although we are going to plug in $\mathbf{y}=(1,1,\dots,1)$, we may compute the rational function $\theta_{\mathcal{K}}(\mathbf{x},\mathbf{y})$ by evaluating it in any sufficiently large domain, such as the one used here.

Analogously, we have for the open generating function

$$\sigma_{\mathcal{K}^\circ}(\mathbf{x})=\operatorname{const}\left(\frac{x_1\mathbf{z}^{\mathbf{a}_1}}{1-x_1\mathbf{z}^{\mathbf{a}_1}}\frac{x_2\mathbf{z}^{\mathbf{a}_2}}{1-x_2\mathbf{z}^{\mathbf{a}_2}}\cdots\frac{x_d\mathbf{z}^{\mathbf{a}_d}}{1-x_d\mathbf{z}^{\mathbf{a}_d}}\frac{z_1}{1-z_1}\frac{z_2}{1-z_2}\cdots\frac{z_m}{1-z_m}\right)$$

and we define

$$\theta_{\mathcal{K}^\circ}(\mathbf{x},\mathbf{y})=\operatorname{const}_{\mathbf{z}}\left(\prod_{j=1}^{d}\frac{x_j\mathbf{z}^{\mathbf{a}_j}}{1-x_j\mathbf{z}^{\mathbf{a}_j}}\prod_{k=1}^{m}\frac{y_kz_k}{1-y_kz_k}\right).$$

Then $\sigma_{\mathcal{K}^\circ}(\mathbf{x})=\theta_{\mathcal{K}^\circ}(\mathbf{x},1,1,\dots,1)$ and for $|\mathbf{x}|,|\mathbf{y}|<1$

$$\theta_{\mathcal{K}^\circ}(\mathbf{x},\mathbf{y})=\int\prod_{j=1}^{d}\frac{x_j\mathbf{z}^{\mathbf{a}_j}}{1-x_j\mathbf{z}^{\mathbf{a}_j}}\prod_{k=1}^{m}\frac{y_kz_k}{1-y_kz_k}\frac{d\mathbf{z}}{\mathbf{z}}\ .$$

The integral representations of $\theta_{\mathcal{K}}$ and $\theta_{\mathcal{K}^\circ}$ now suggest how to prove Theorem 1—make a change of variables $\mathbf{z}\to 1/\mathbf{z}$, say in $\theta_{\mathcal{K}^\circ}$:

$$\theta_{\mathcal{K}^\circ}(\mathbf{x},\mathbf{y})=\int\prod_{j=1}^{d}\frac{x_j\mathbf{z}^{-\mathbf{a}_j}}{1-x_j\mathbf{z}^{-\mathbf{a}_j}}\prod_{k=1}^{m}\frac{y_kz_k^{-1}}{1-y_kz_k^{-1}}\frac{d\mathbf{z}}{\mathbf{z}}\ .$$

Hence the rational function $\theta_{\mathcal{K}^\circ}(1/\mathbf{x},1/\mathbf{y})$ has the integral representation

$$\begin{aligned}\theta_{\mathcal{K}^\circ}(1/\mathbf{x},1/\mathbf{y})&=\int\prod_{j=1}^{d}\frac{x_j^{-1}\mathbf{z}^{-\mathbf{a}_j}}{1-x_j^{-1}\mathbf{z}^{-\mathbf{a}_j}}\prod_{k=1}^{m}\frac{y_k^{-1}z_k^{-1}}{1-y_k^{-1}z_k^{-1}}\frac{d\mathbf{z}}{\mathbf{z}}\\&=\int\prod_{j=1}^{d}\frac{1}{x_j\mathbf{z}^{\mathbf{a}_j}-1}\prod_{k=1}^{m}\frac{1}{y_kz_k-1}\frac{d\mathbf{z}}{\mathbf{z}}\\&=(-1)^{d+m}\int\prod_{j=1}^{d}\frac{1}{1-x_j\mathbf{z}^{\mathbf{a}_j}}\prod_{k=1}^{m}\frac{1}{1-y_kz_k}\frac{d\mathbf{z}}{\mathbf{z}}\ ,\end{aligned}$$

valid for $|\mathbf{x}|,|\mathbf{y}|>1$. It remains to prove that the rational function given by the integral

$$\int\frac{1}{\prod_{j=1}^{d}\left(1-x_j\mathbf{z}^{\mathbf{a}_j}\right)\prod_{k=1}^{m}\left(1-y_kz_k\right)}\frac{d\mathbf{z}}{\mathbf{z}}\tag{2}$$

with $|\mathbf{x}|,|\mathbf{y}|<1$ equals the rational function given by the integral

$$(-1)^m\int\frac{1}{\prod_{j=1}^{d}\left(1-x_j\mathbf{z}^{\mathbf{a}_j}\right)\prod_{k=1}^{m}\left(1-y_kz_k\right)}\frac{d\mathbf{z}}{\mathbf{z}}\tag{3}$$

with $|\mathbf{x}|, |\mathbf{y}| > 1$, since once we have this equality, we have $\theta_{\mathcal{K}^\circ}(1/\mathbf{x}, 1/\mathbf{y}) = (-1)^d \theta_{\mathcal{K}}(\mathbf{x}, \mathbf{y})$; plugging in $\mathbf{y} = (1, 1, \dots, 1)$ (whereupon $1/\mathbf{y} = (1, 1, \dots, 1)$ as well) shows that $\theta_{\mathcal{K}^\circ}(1/\mathbf{x}) = \theta_{\mathcal{K}}(\mathbf{x})$ as desired.

Let us consider as the innermost integral, the one with respect to z_1. Almost all of the poles of the integrand $f(z_1)$ are at the solutions z_1 of the equations $1 - x_j \mathbf{z}^{\mathbf{a}_j} = 0$ (for $j = 1, \dots, d$) and $1 - y_1 z_1 = 0$. Since $|z_2|, |z_3|, \dots, |z_m| = 1$, each z_1-pole is inside or outside the unit circle, depending on the exponent of z_1 in $\mathbf{z}^{\mathbf{a}_j}$ and on whether a given x_j or y_j has magnitude smaller or larger than 1. But this means that the z_1-integrals in (2) and (3) pick up the residues of complementary singularities.

The only other potential poles are at zero and infinity, induced by the extra factor of $\frac{1}{z_1}$. We claim that there are no residues at these poles, if they exist. Indeed, as z_1 approaches zero, the residue $z_1 f(z_1)$ is the product of the factors $\frac{1}{1 - x_j \mathbf{z}^{\mathbf{a}_j}}$ and $\frac{1}{1 - y_k z_k}$. Since K is in the nonnegative orthant, in each inequality at least one coefficient must be negative; therefore, in at least one of the former factors, the exponent of z_1 must be negative. This factor then goes to zero as z_1 does, while the norms of all of the other factors either go to zero (if the exponent of z_1 is negative), one (if the exponent is positive), or a constant (if z_1 does not appear at all.) Therefore, this residue is equal to zero.

A similar argument eliminates the residue at infinity. After a change of variables $z_1 \to \frac{1}{z_1}$, this residue at infinity is (up to sign) the limit of the product of these same factors. However, the factor $\frac{1}{1 - y_1 z_1}$ goes to zero, while all other factors again go to zero, one, or a constant, depending on whether the exponent of z_1 is positive, negative, or zero respectively. This completes the argument that the z_1-integrals in (2) and (3) differ by a minus sign.

We can use the same argument for the next variable if we know that the z_1-integral results in a rational function with a similar-looking denominator as the integrands in (2) and (3). But this follows from Euler's generating function: The z_1-integral in (2) gives the generating function of the cone described by $\mathbf{r} \geq 0$ and the first row inequality in $\mathbf{A}\,\mathbf{r} \leq 0$, in the variables $x_1 (z_2, \dots, z_m)^{\mathbf{a}'_1}, \dots, x_d (z_2, \dots, z_m)^{\mathbf{a}'_d}$ and $\mathbf{y}$, where $\mathbf{a}'_1, \dots, \mathbf{a}'_d$ are the column vectors of $\mathbf{A}$ after we removed the first row. It is not hard to show (from the simplicial case) that the generating function of any such cone has as denominator a product of terms of the form 1 minus a monomial of the variables, so the z_2-integrand has the desired form.

There is a possible obstruction here, which is that these monomials may have x's and y's appearing both in the numerator and the denominator. If we pick a poor choice of norms of the x's and y's, we may not be able to evaluate these integrals, since the norm of this monomial may be 1, in which case there are poles on the unit circle. However, we can get around this simply by looking at an open set where none of these magnitudes are 1 to evaluate the integral for $|\mathbf{x}|, |\mathbf{y}| > 1$, and looking at the image of this set under $\mathbf{x} \to \mathbf{x}^{-1}, \mathbf{y} \to \mathbf{y}^{-1}$ to evaluate the integral (of the same rational function) for $|\mathbf{x}|, |\mathbf{y}| < 1$.

We also need to show that the residues of the intermediate integrals at zero and infinity are both zero. As we evaluate the first integral, we obtain the sum of residues at poles, each of which has z_1 equal to a monomial in the other variables, possibly with fractional powers. (If fractional powers bother you, simply replace each variable by a power of itself.) The corresponding residue is obtained by plugging this monomial in for z_1 in all of the other factors, along with the remaining portion of

the factor which the pole was extracted out of. Each of these residues has denominator a product of terms of the form 1 minus a monomial of the variables. As we evaluate each integral, we perform further substitutions of monomials in the x's, y's, and later z's for each z-variable in succession.

As z_i goes to zero or infinity, each factor in each residue goes to 1, 0, or a constant depending on whether the exponent of z_i is positive, negative, or zero, as in the first integral. We need to show that in each residue, one of the factors goes to 0. As z_i goes to infinity, the factor $\frac{1}{1-y_i z_i}$, which is unblemished (since it has no other z-variables), goes to zero. The analysis when z_i goes to 0 is a bit trickier. For each previous z-variable, representing an equality, we have picked one of the factors, corresponding to one x or y-variable. Substituting for that variable amounts to solving the given equality to express that variable in terms of the other variables, and plugging the expression into the other equalities to create a new system of equations. The residue will be given by the Euler-type generating function of this new system of equalities.

With this formulation, it is clear that if we have a solution $\begin{pmatrix}\mathbf{x}\\ \mathbf{y}\end{pmatrix}$, then the same vector with the deleted variable removed will be a solution of the new system of equalities; see Example 3. Taking a vector with all positive entries and following it through the process of eliminating the first $i-1$ rows by deleting the corresponding columns, we find that the system of equalities at the i-th step has an all-positive solution. From this, it follows that one of the entries in each row must be negative, and in particular there must be a negative number in the i-th row. The corresponding factor of the denominator will have z_i appearing with negative exponent, and thus will go to 0 as z_i goes to 0. So each summand has residue 0 as z_i goes to 0, and thus the entire i-th integrand does.

Therefore, for each integral, we have complementary residues counted in integrals (2) and (3), introducing a minus sign; after factoring this out, the two integrals produce identical rational functions which move on to the next integral. Since we have m integrals, we obtain that the integrals in (2) and (3) differ by a factor of $(-1)^m$ as desired. This completes the proof.

It is worth noting that we made the decision to replace the $\frac{1}{1-x_i\mathbf{z}^{\mathbf{a}_i}}$ factors in $\theta_{\mathcal{K}}(\mathbf{x},\mathbf{y})$ by factors of $\frac{x_i\mathbf{z}^{\mathbf{a}_i}}{1-x_i\mathbf{z}^{\mathbf{a}_i}}$ in $\theta_{\mathcal{K}^\circ}(\mathbf{x},\mathbf{y})$. Since the facets correspond to the y-variables, we do not need strict inequalities on the x-coordinates in order to compute the generating function of $\mathcal{K}^\circ$. Instead, we chose to use this expansion, since it yields the correct complementarity statement regarding the poles.

3. An illustrative example

In this section, we give an example illustrating the proof of the previous section.

Example 3. *Let $\mathcal{K}$ be the pyramidal cone in the positive orthant given by the inequalities:*

$$\begin{aligned} x_1 - x_2 &\leq 0 \\ x_1 - x_3 &\leq 0 \\ x_2 - 2x_1 &\leq 0 \\ x_3 - 2x_1 &\leq 0. \end{aligned}$$

This is a cone with vertex $(0,0,0)$ over a square in the $x_1 = 1$ plane with vertices $(1,i,j)$ for $i,j \in \{1,2\}$.

The matrix A is then $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$; the modified matrix $A' := (A|I)$ is

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the original integrand (to be integrated over $z_1, \dots, z_4$) is:

$$\frac{1}{(1 - x_1 z_1 z_2 z_3^{-2} z_4^{-2})(1 - x_2 z_1^{-1} z_3)(1 - x_3 z_2^{-1} z_4)(1 - y_1 z_1)(1 - y_2 z_2)(1 - y_3 z_3)(1 - y_4 z_4)}.$$

We illustrate what happens when we compute a residue in an intermediate integral. Suppose we are in the process of computing the residue with respect to z_1 at the pole corresponding to the first factor. This amounts to substituting $z_1 = x_1^{-1} z_2^{-1} z_3^2 z_4^2$ into all of the other factors. Consider the second factor, $\left(1 - x_2 z_1^{-1} z_3\right)$. This becomes $\left(1 - x_2 \left(x_1 z_2 z_3^{-2} z_4^{-2}\right) z_3\right)$; a moment's reflection will reveal that this is equivalent to adding an appropriate multiple of the first column of A' (here, 1) to the second column to cancel out the element in the first row.

So, in effect, this residue is the Euler-type generating function of the matrix obtained by doing Gaussian elimination, adding a multiple of the first column (since we picked the x_1 term) to all other columns to eliminate the first row (since we are integrating over z_1), and then deleting the first column. The only difference is that there are a few x_1's thrown into each term, which are irrelevant for our purposes; recall that the point of this statement in the proof was merely to show that in this residue, some term exists with a negative power of each z_i. It suffices to show that this new matrix has a negative entry in each row.

But, as noted in the proof, if we pick a positive solution to the original equation $A' \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = 0$, say $(x_1, x_2, x_3, y_1, y_2, y_3, y_4) = (2, 3, 3, 1, 1, 1, 1)$, then if we remove x_1, the new tuple $(x_2, x_3, y_1, y_2, y_3, y_4) = (3, 3, 1, 1, 1, 1)$ will be a solution of the new matrix, which in this case is

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 1 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

That the elimination procedure has this property is easily verified; it comprises using the eliminated equation to solve for the eliminated variable in terms of the other variables, then plugging that into the other equations.

Since there is an all-positive solution to the new set of equations, there must be at least one negative entry in each row.

4. Extensions and applications

An interesting extension of Theorem 1 is the following: Let

$$\mathcal{K}_1 = \left\{ \mathbf{r} \in \mathbb{R}^d_{\geq 0} : \; \mathbf{A}\,\mathbf{r} \leq 0 \text{ and } \mathbf{B}\,\mathbf{r} < 0 \right\}$$

and

$$\mathcal{K}_2 = \left\{ \mathbf{r} \in \mathbb{R}^d_{> 0} : \; \mathbf{A}\,\mathbf{r} < 0 \text{ and } \mathbf{B}\,\mathbf{r} \leq 0 \right\},$$

that is, $\mathcal{K}_1$ and $\mathcal{K}_2$ are *half-open* cones whose constraints are defined by the same matrix, however, those facets (codimension-1 faces) which are contained in $\mathcal{K}_1$ are not in $\mathcal{K}_2$ and vice versa. Since by assumption $(0, \dots, 0)$ is the only intersection point of the fully closed cone with the coordinate hyperplanes, at most one of $\mathcal{K}_1$ and $\mathcal{K}_2$ intersects these hyperplanes, so the domains are accurate.

Theorem 4. *As rational functions, if the set of facets closed in $\mathcal{K}_1$ is contractible, $\sigma_{\mathcal{K}_1}(1/\mathbf{x}) = (-1)^d \, \sigma_{\mathcal{K}_2}(\mathbf{x})$.*

Stanley's original proof [7] showed that this was true whenever the set of facets consists of those visible from a certain point outside the cone; shelling-based proofs of Theorem 1 prove this theorem whenever the set of closed facets is contractible.

Our proof technique is easily adjusted to this more general setting.

Suppose $\mathbf{A} \in \mathbb{Z}^{d \times m}$ has columns $\mathbf{a}_1, \dots, \mathbf{a}_d$, and $\mathbf{B} \in \mathbb{Z}^{d \times n}$ has columns $\mathbf{b}_1, \dots, \mathbf{b}_d$, then Lemma 2 gives

$$\sigma_{\mathcal{K}_1}(\mathbf{x}) = \operatorname{const}_{\mathbf{z},\mathbf{w}} \left(\frac{1}{(1 - x_1 \mathbf{z}^{\mathbf{a}_1} \mathbf{w}^{\mathbf{b}_1}) \cdots (1 - x_d \mathbf{z}^{\mathbf{a}_d} \mathbf{w}^{\mathbf{b}_d})} \frac{1}{(1 - z_1) \cdots (1 - z_m)} \frac{w_1}{1 - w_1} \cdots \frac{w_n}{1 - w_n} \right)$$

and

$$\sigma_{\mathcal{K}_2}(\mathbf{x}) = \operatorname{const}_{\mathbf{z},\mathbf{w}} \left(\frac{x_1 \mathbf{z}^{\mathbf{a}_1}}{1 - x_1 \mathbf{z}^{\mathbf{a}_1} \mathbf{w}^{\mathbf{b}_1}} \cdots \frac{x_d \mathbf{z}^{\mathbf{a}_d} \mathbf{w}^{\mathbf{b}_d}}{1 - x_d \mathbf{z}^{\mathbf{a}_d}} \frac{z_1}{1 - z_1} \cdots \frac{z_m}{1 - z_m} \frac{1}{(1 - w_1) \cdots (1 - w_n)} \right).$$

The proof that

$$\theta_{\mathcal{K}_1}(\mathbf{x}, \mathbf{y}) = \int \prod_{j=1}^{d} \frac{1}{1 - x_j \mathbf{z}^{\mathbf{a}_j} \mathbf{w}^{\mathbf{b}_j}} \prod_{k=1}^{m} \frac{1}{1 - y_k z_k} \prod_{i=1}^{n} \frac{y_{m+i} w_i}{1 - y_{m+i} w_i} \frac{d\mathbf{z}}{\mathbf{z}} \frac{d\mathbf{w}}{\mathbf{w}}$$

and

$$\theta_{\mathcal{K}_2}(\mathbf{x}, \mathbf{y}) = \int \prod_{j=1}^{d} \frac{x_j \mathbf{z}^{\mathbf{a}_j} \mathbf{w}^{\mathbf{b}_j}}{1 - x_j \mathbf{z}^{\mathbf{a}_j} \mathbf{w}^{\mathbf{b}_j}} \prod_{k=1}^{m} \frac{y_k z_k}{1 - y_k z_k} \prod_{i=1}^{n} \frac{1}{1 - y_{m+i} w_i} \frac{d\mathbf{z}}{\mathbf{z}} \frac{d\mathbf{w}}{\mathbf{w}},$$

both defined for $|\mathbf{x}|, |\mathbf{y}| < 1$, satisfy a Stanley-type reciprocity identity proceeds along the exact same lines as our proof of Theorem 1. Note that as per the comment at the end of the previous section, even though the original cones $\mathcal{K}_1$ and $\mathcal{K}_2$ are partially open, we can arbitrarily choose to "invert" all of the x-variables in $\mathcal{K}_2$ and none in $\mathcal{K}_1$ (since $\mathcal{K}_2$ contains no points with any x-coordinate equal to zero.)

However, this proof cannot work in all cases, since the generalized reciprocity theorem is not true in all cases. What can go wrong is that in the partially eliminated matrix, the term (or, rather, all

of the terms) with positive w_i-exponent can be those of the form $\frac{y_{m+i}w_i}{1-y_{m+i}w_i}$, which does not in fact go to zero as w_i goes to infinity. It would be interesting to come up with an elegant charcterization of when this happens, which would provide an elegant proof of the generalized Stanley theorem for a subset of half-open cones.

A particular nice application of Theorem 1 concerns the counting function $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbb{Z}^d\right)$ for a rational convex polytope $\mathcal{P}$, that is, the convex hull of finitely many points in $\mathbb{Q}^d$. Ehrhart proved in [1] the fundamental structural result about $L_{\mathcal{P}}(t)$, namely that it is a quasi-polynomial in t (for a definition and nice discussion of quasi-polynomials see [9, Chapter 4]). Ehrhart conjectured and partially proved the following reciprocity theorem, which was proved by Macdonald [5].

Theorem 5 (Ehrhart-Macdonald). *The quasi-polynomials $L_{\mathcal{P}}$ and $L_{\mathcal{P}^\circ}$ satisfy*

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t) \ .$$

5. Concluding remarks

1. As already mentioned, most proofs of Theorems 1 and 5 use shellings of a polyhedron or finite additive measures (see, e.g., [2, 5, 6]). The only exceptions we are aware of are proofs via complex analysis (e.g., as above or in [7]) and commutative algebra (see, e.g., [8, Section I.8]), as well as a recent proof [4] by the first author and Frank Sottile using irrational decomposition.

2. It is a fun exercise to deduce Theorem 5 from Theorem 1, for example by considering the generating function of the $(d+1)$-cone generated by $(\mathbf{v}_1, 1), \dots, (\mathbf{v}_n, 1)$, where $\mathbf{v}_1, \dots, \mathbf{v}_n$ are the vertices of $\mathcal{P}$, applying Stanley reciprocity, and then specializing the rational generating functions by setting the first d variables to 1.

3. Finally, there exists an extension of Theorem 5 corresponding to Theorem 4: one includes some of the facets of the polytope on one side, and the complementary set of facets on the other side.

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